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Space singularity and intrinsic quantum curvature in charged massive non-rotating Reissner–Nordström black hole

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Abstract Recent research has revealed that curvatures that are primarily orthogonal to the conventional semiclassical type may emerge when a geometric quantization ansatz is employed in Einsteinian general relativity. The timelike geodesic congruence in Reissner–Nordström metric, which has been both analytically derived and numerically examined, is used to investigate whether these phenomena are not artifacts of specific coordinate systems. It has been observed that the geodesic congruence expansion remains consistently non-vanishing, especially at small radial distance r. As r decreases, a significantly exponential evolution is found. The finiteness of the Kretschmann scalar for both conventional and quantized metric enables a definitive evaluation, suggesting that the conventional as well as the quantum curvatures are likely real, essential, and intrinsic (not artifacts of certain coordinate systems). Furthermore, we conclude that the proposed quantization approach seems to enhance the local focusing of the quantum curvatures in the charged, massive, non-rotating, and spherically symmetric Reissner–Nordström black hole. We also find that the quantum curvatures, despite their approximate qualitative estimation, indicate a complex spacetime structure that appears to be overlooked due to the semiclassical approximation utilized in conventional general relativity.

1 Introduction

At relativistic energies (quantum scales), spacetime is likely to abandon its continuous and smooth texture [1–4]. The fine-structured discretization, roughness, foaming, nonlocalizing, and fluctuating aspects emerge as the prevailing features [5,6]. Over the course of the last century, the challenge of realizing the quantization of spacetime or unifying the principles of general relativity (GR) and quantum mechanics (QM) continues to be unanswered. A few years ago, one of the authors (AT) introduced a canonical geometric quantization approach that focuses on the fundamental metric in GR and thereby the underlying spacetime [7–16]. Consequently, a revision inspired by QM was implemented and an extended GR could be formulated.

The idea here is ground on generalizations to be imposed on both theories. The noncommutative relations in QM [17,18], the Heisenberg uncertainty principle, are generalized to account for finite gravitational fields [19–21]. On the other hand, the pseudo-Riemann geometry in GR is to be substituted with Finsler geometry, characterized by dimensions that include both coordinates and velocities. Additionally, the kinematics of free-falling quantum particles is considered in the additional curvatures that emerge. The additional curvatures are distinctly orthogonal to the conventional ones which obviously exclusively manifest sources of conventional gravity, while the additional curvatures are quantum by definition and likely refer to sources of quantum gravity. This manuscript focuses on uncovering nature of the new type of curvatures by investigating whether they are real, essential, and intrinsic or simply badly artifacts in particular coordinate systems.

The generalizations that are applied to both fundamental theories make it possible to incorporate quantum mechanical features into extended differential geometry, whether it is Finsler or Hamilton geometry. The relevant metric can be derived using Hessian matrix. Details about the proposed geometric quantization, its implementations across different

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phenomena, particularly where conventional GR experiences notable challenges, and potential paths for improvement, can be found in Refs. [7–16]. To summarize, the emerged quantum geometry, which incorporates diverse geometric elements from the symplectic geometries, phase-space manifolds, reveals intrinsic, real and essential additional curvatures, along with affine connections, curvature tensors, and geometric structures [7–16]. As mentioned briefly, the kinematic theory applicable to a free-falling quantum particle possessing a positive mass m establishes a comprehensive framework for the implementation of quantum-mechanical principles on the tangent (Finsler) or cotangent (Hamilton) manifolds. Specifically, the traditional spacetime in pseudo-Riemann geometry is extended to incorporate Finsler or Hamilton structures, which are defined by auxiliary four-vectors x_0^μ and velocities $\dot{x}_0^\nu$ or x_0^μ and momenta $\dot{p}_0^\nu$.

The concept introduced to QM is known as the relativistic generalized uncertainty principle (RGUP), which modifies the noncommutative relations [22–24]. Thus, the auxiliary momentum operator $\hat{p}_0^\mu$ is generalized to $\phi(p_0)\hat{p}_0^\mu$. The function $\phi(p_0) = 1 + \beta p_0^\rho p_{0\rho}$ is essential, where ρ is dummy index and β is a parameter that can be determined at least its upper bound, through empirical and observational means [25]. To begin with, the Finsler metric can be readily extracted from the Hessian matrix of the corresponding structure $F^2(x_0^\mu, \dot{x}_0^\nu$, where the homogeneity in $\dot{x}_0^\nu$) for a quantum particle with positive mass results in $F^2(x_0^\mu, m\dot{x}_0^\nu) = mF^2(x_0^\mu, \dot{x}_0^\nu) \equiv F^2(x_0^\mu, p_0^\nu)$. For RGUP, one finds that $F^2(x_0^\mu, p_0^\nu) \rightarrow F^2(x_0^\mu, \phi(p_0)p_0^\nu)$, i.e., $m\dot{x}^\mu \equiv p_0^\mu$, which facilitates the derivation of a metric that includes quantum-mechanical modifications.

This specific quantization ansatz, which targets exclusively the fundamental metric, the principle geometric quantity, reveals that all the postulates of Einsteinian GR are presumably preserved. The resulting metric tensor $\tilde{g}_{\mu\nu}$, particularly when it is expressed as a conformal transformation of its standard counterpart $g_{\mu\nu}$, appears to uphold the symmetric property and also shares the invariance characteristic with the Weyl tensor under the conformal transformation to $g_{\mu\nu}$. On the other hand, with $\tilde{g}_{\mu\nu}$, the reach of GR is significantly extended, allowing for predictions to be made in both relativistic quantum as well as semiclassical regimes.

This research employs the geometric quantization approach to the timelike geodesic congruence [4,26–31] in Reissner–Nordström black hole [32–39]. The main focus is on the examination of space singularity using $\tilde{g}_{\mu\nu}$. By performing a detailed comparison with the results obtained from the conventional metric tensor $g_{\mu\nu}$, one can identify the exclusive contribution that the proposed geometric quantization introduces. Therefore, an analysis can be conducted to determine if the suggested quantization diminishes or regulates the space singularity. By examining the evolution of a set of trajectories, the congruence of the trajectories is characterized by flow lines produced by velocity fields [26,40]. Another pillar of this study focuses on the nature of the additional curvatures that manifest in spacetime as a result of $\tilde{g}_{\mu\nu}$. To determine if these additional curvatures, which as mentioned are distinguishable from the conventional curvatures that manifest the well-know GR gravity, are not simply artifacts of certain coordinate systems, whether they are known or unknown, and to confirm their real, essential, and intrinsic nature, we perform a thorough examination of the Kretschmann invariant scalar [41,42].

The present manuscript is organized as follows. The formalism is presented in Sect. 2. The timelike geodesic congruence involving $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is detailed in Sects. 2.1.1 and 2.1.2, respectively. The formalism concerning the Kretschmann invariant scalar is provided in Sect. 2.2. Section 3 addresses the numerical analysis, where the timelike geodesic congruence with $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ is discussed in Sect. 3.1, and the findings related to the Kretschmann invariant scalar are examined in Sect. 3.2. Finally, Sect. 4 is reserved for the conclusions.

2 Formalism

The main motivation for the canonical geometric quantization approach [7–16] can be found in the assumption that the metric tensor holds all relevant information about the fundamental spacetime and thereby the underlying geometry. Thus, quantization of metric tensor is considered the suitable approach for quantizing GR

$$\tilde{g}_{\mu\nu} = C(x, p)\, g_{\mu\nu}, \tag{1}$$

where the conformal coefficient $C(x, p)$ includes all components imposed by QM that are applied to the quantized metric tensor. Formulating $C(x, p)$ with precision presents a mathematical challenge. While attempting to resolve this challenge, the expression indicated in Eq. (1) is an approximation that, for the time being, appears to be unavoidable. In this regard, let us recall the non-truncated expression for $\tilde{g}_{\mu\nu}$,

$$\tilde{g}_{\mu\nu} = \left(\phi^2 + 2\frac{\kappa}{(p_0^0)^2}F^2\right)\left[1 + \frac{\dot{p}_0^\mu \dot{p}_0^\nu}{\mathscr{F}^2}\left(1 + 2\beta p_0^\rho p_{0\rho}\right)\right] g_{\mu\nu} + \left[\frac{dx_0^\mu}{d\zeta^\mu}\frac{dx_0^\nu}{d\zeta^\nu} + \left(1 + 2\beta p_0^\rho p_{0\rho}\right)\frac{dp_0^\mu}{d\zeta^\mu}\frac{dp_0^\nu}{d\zeta^\nu}\right] d_{\mu\nu}, \tag{2}$$

where $\mathscr{F}$ denotes the maximum proper gravitational force that allows the quantum particle to achieve its highest gravitational acceleration during its motion in the additional curvatures; the sources of quantum gravity [43–45]. The discovery of $\mathscr{F}$ as a new physical constant was reported in Ref. [46], for example. Furthermore, $\dot{p}_0^\mu \dot{p}_0^\nu$ are the squared gravitational force acting on that quantum particle. In this context, ζ^{μ} serves as parameterizations that connect the coordinates in the Finsler tangent or Hamilton cotangent bundle to the pseudo-Riemann coordinates. This restriction might be relaxed when one succeeds in deriving the phase-space metric tensor, the four-dimensional spacetime that is merged in four-dimensional momentum space [47]. Given that the second line in Eq. (2) cannot be evaluated in terms of $g_{\mu\nu}$ as is the case with the first line, it was assumed that the first line becomes the dominant one so that

$$C(x,p) = \left(\phi^2 + 2\frac{\kappa}{(p_0^0)^2}F^2\right)\left[1 + \frac{\dot{p}_0^{\mu}\,\dot{p}_0^{\nu}}{\mathscr{F}^2}\left(1 + 2\beta p_0^{\rho} p_{0\rho}\right)\right], \tag{3}$$

The presumption that the second line of Eq. (2) ought to be zero is a simplistic approximation. In this regard, let us express $d_{\mu\nu}$,

$$\begin{aligned} d_{\mu\nu} = \frac{4\kappa F^2}{(p_0^0)^2}\Big[&(p_0^0)^2\,(3\,\phi(p_0)-1)\;\ell_{\mu}\,\ell^{\sigma}\,g_{\sigma\mu} \\ &-F\,p_0^0\,(3\,\phi(p_0)-1)\;\ell^{\sigma}\,\left[\delta_{0\nu}\,g_{\sigma\mu}+\delta_{0\mu}\,g_{\sigma\nu}\right] \\ &+2\,F^2\,(5\,\phi(p_0)-2)\;\delta_{0\mu}\,\delta_0^{\sigma}\,g_{\sigma\nu}\Big]. \end{aligned} \tag{4}$$

where $\ell_{\mu} := \partial F/\partial \dot{x}_0^{\mu}$ and $\kappa = \beta/(p_0^0)^2$.

Let us now discuss the approximation $d_{\mu\nu} \rightarrow 0$ that has been imposed in Eq. (1). There are strict mathematical limitations against it including that the RGUP function $\phi(p_0)$ does not depend either on either x_0 or $d_{\mu\nu}$. Furthermore, from a physical perspective, $\phi(p_0)$ must be finite, while the assumption that $d_{\mu\nu} \rightarrow 0$ apparently assumes that $\phi(p_0) \rightarrow 0$, as well. In light of the complexity associated with $d_{\mu\nu}$, namely reexpressing it in terms in $g_{\mu\nu}$,. The truncation of Eq. (2), notwithstanding the mathematical and physical limitations, seems to enable the retention of some quantum-mechanical revisions. As this approach represents a new quantization solely based on the fundamental metric, it is essential to validate its accuracy and effectiveness even partly. In this regard, qualitative analyses, such as the one presented here, are not merely defined by accurate estimations of the quantum operators in Eq. (2). Whatever the numerical intentions may be, the numerical estimation of quantum operators likely undergo multiple approximations. This implies that even if the full Eq. (2) is considered, the measure, numerical analysis, of the quantum operators are subject to various approximations. Furthermore, the current numerical analysis is based on the average values of quantum quantities and operators in $C(x,p)$, Eq. (3). This permits the consideration of $C(x,p)$ as a correction to $g_{\mu\nu}$, the conformal coefficient, which evidently indicates that $\tilde{g}_{\mu\nu} \equiv g_{\mu\nu}$ is obviously maintained when $C(x,p) = 1$. In summary, it is important to highlight that the values of $C(x,p)$, whatever they may be, are universally positive, and the final conclusions are qualitatively valid. The averaged values that consider $C(x,p) \in]0,1]$ seem to provide noteworthy insights into the proposed geometric quantization of metric and spacetime and thereby the underlying geometry. It is important to note that the present calculations treat $C(x,p)$ as a free parameter, serving as an indicator for the values of $C(x,p)$ where additional curvatures and sources of an alternative type of gravity, probably quantum, emerge. Additionally, the space singularity will be reassessed in relation to the charged, massive, non-rotating, spherically symmetric Reissner–Nordström black hole. Comparable analyses have been carried out on different types of black holes [8–10,12]. Various publications, especially Ref. [7], provide detailed insights into the geometric quantization. The present manuscript concentrates on the timelike geodesic congruence and its association with $C(x,p)$, as shown in Fig. 1. The geodesic equations are outlined in Appendix A. An elaboration on the parametric study can be found in the Appendix B.

Section 2.1.1 presents the analytical expressions for the timelike geodesic congruence [14] derived using $g_{\alpha\beta}$. The related expressions obtained with $\tilde{g}_{\alpha\beta}$ are outlined in Sect. 2.1.2.

2.1 Timelike geodesic congruence

The static solution of the Einstein–Maxwell field equations, $ds^2 = -c^2 d\tau^2$ with τ is the proper time, in the gravitational field of charged, massive, non-rotating, and spherically symmetric Reissner–Nordström black hole is given as [48–53],

$$\begin{aligned} ds^2 = &-\left(1-\frac{2GM}{c^2 r}+\frac{GQ^2}{4\pi\varepsilon c^4 r^2}\right)c^2 dt^2 \\ &+\left(1-\frac{2GM}{c^2 r}+\frac{GQ^2}{4\pi\varepsilon c^4 r^2}\right)^{-1} dr^2 + r^2 d\Omega^2, \end{aligned} \tag{5}$$

where M is the mass, Q is an electric charge, r represents the radial distance, while t denotes the time coordinate, which is measured by a stationary clock located at infinity. The quantity $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ represents the standard line element on the surface of a two-sphere, where (θ,ϕ) are the spherical angles. The quantity $\varepsilon = (\mu_0 c^2)^{-1}$ represents the electric constant, where c is the speed of light and μ_0 is the vacuum permeability or magnetic constant. Let us assume geometrized units, i.e., $c = G = 1$.

The timelike geodesic congruence with conventional and quantized metric tensor are derived in Sects. 2.1.1 and 2.1.2, respectively

2.1.1 Timelike geodesic congruence with $g_{\mu\nu}$

In comoving coordinates, $x^{\mu} = (t, r, \theta, \phi)$, the first geodesic equation can be expressed as

$$\frac{du^t}{d\tau} = \frac{2(Q^2 - 4\pi M\varepsilon r)u^t}{r[Q^2 + 4\pi\varepsilon r(r-2M)]}\frac{dr}{d\tau}. \tag{6}$$

The solution for u^t reads

$$u^t = \frac{r^2}{Q^2 + 4\pi\varepsilon r(r-2M)]}. \tag{7}$$

When the third and fourth components of the velocity field, u^θ and u^ϕ, are vanishing, and considering the normalization condition for the finite components of the velocity field, $u^t u^r g_{tr} = -1$, the second geodesic equation can be obtained

$$\begin{aligned}\frac{du^r}{d\tau} &= \frac{1}{2}\Delta(r)\left[\frac{(Q^2 - 4\pi\varepsilon M r)}{2\pi\varepsilon r^3}\left(u^t\right)^2\right. \\ &\quad - \frac{8\pi\varepsilon r(Q^2 + 4\pi\varepsilon M r)}{\left[(Q^2 + 4\pi\varepsilon r(r-2M)\right]^2}\left(u^r\right)^2 \\ &\quad \left. + 2r\left[\left(u^\theta\right)^2 + \sin^2\theta\left(u^\phi\right)^2\right]\right],\end{aligned} \tag{8}$$

where $\Delta(r) = 4\pi\varepsilon r^2 - 8\pi\varepsilon M r + Q^2$. Equation (8) can be solved as

$$\begin{aligned}u^r &= \frac{1}{4}\left[\frac{r^2 - 4\pi\varepsilon Q^2 - 16\pi\varepsilon^2 r(r-2M)}{\pi\varepsilon\left[Q^2 + 4\pi\varepsilon r(r-2M)\right]}\right]^{1/2} \\ &\quad \times \left[4 + \frac{Q^2 - 8\pi\varepsilon M r}{\pi\varepsilon r^2}\right]^{1/2}.\end{aligned} \tag{9}$$

For these two finite velocity fields, the expansion of the geodesic congruence, $\Theta = u^\alpha_{;\alpha} = u^\alpha_{,\alpha} + u^\sigma \Gamma^\alpha_{\sigma\alpha}$ can be derived in the gravitational field of Reissner–Nordström black hole

$$\Theta = \frac{r^2 - 2\pi\varepsilon Q^2 + 8\pi\varepsilon^2 r(3M - 2r)}{2\pi\varepsilon^{3/2} r^2\left[Q^2 + 4\pi\varepsilon(r-2M)\right]^{1/2}\left[\frac{r^2 - 4\pi\varepsilon Q^2 - 16\pi\varepsilon^2 r(r-2M)}{\pi\varepsilon\left[Q^2 + 4\pi\varepsilon(r-2M)\right]}\right]^{1/2}}. \tag{10}$$

Then, the evolution of the geodesic congruence expansion, Eq. (10), can be obtained

$$\begin{aligned}\frac{d\Theta}{d\tau} &= -\frac{1}{8\pi\varepsilon^2 r^4\left[r^2 - 4\pi\varepsilon Q^2 - 16\pi\varepsilon^2 r(r-2M)\right]} \\ &\quad \times \left\{r^4 - 6\pi\varepsilon Q^2 r^2 + 16\pi\varepsilon^2\left(Q^4 + 4Mr^3 - 2r^4\right)\right. \\ &\quad + 32\pi\varepsilon^3 Q^2 r(3r - 8M) \\ &\quad \left. + 128\pi\varepsilon^4 r^2(9M^2 - 8Mr + 2r^2)\right\}.\end{aligned} \tag{11}$$

The subsequent section introduces the timelike geodesic congruence derived from the quantized tensor tensor within the gravitational field of Reissner–Nordström black hole.

2.1.2 *Timelike geodesic congruence with $\tilde{g}_{\mu\nu}$*

Also here, we initiate our discussion with comoving coordinates. The two finite components of the velocity fields are u^t and u^r. By employing the quantized metric $\tilde{g}_{\mu\nu}$, we finite that $\tilde{u}^t$ and $\tilde{u}^r$ are the quantized finite counterparts,

$$\tilde{u}^t = \exp\left[2\ln(r) - \ln(\Delta(r)) - \frac{r}{C(x,p)}F^2_{,\gamma}\right], \tag{12}$$

$$\tilde{u}^r = \frac{1}{r}\left[r^2\exp\left(-\frac{2rF^2_{,\gamma}}{C(x,p)}\right) - \frac{\Delta(r)}{C(x,p)}\right]^{1/2}, \tag{13}$$

where $F^2_{,\gamma}$ is the comma derivative of the Finsler/Hamilton structure, Eq. (A5). A comparable algebraic procedure, as applied in the preceding section, results in the associated expansion of the quantized geodesic congruence, $\tilde{\Theta} = \tilde{u}^\alpha_{;\alpha} = \tilde{u}^\alpha_{,\alpha} + \tilde{u}^\sigma \tilde{\Gamma}^\alpha_{\sigma\alpha}$. Then,

$$\begin{aligned}\tilde{\Theta} &= \frac{\exp\left(-\frac{2rF^2_{,\gamma}}{C(x,p)}\right)}{2C(x,p)^{3/2} r^2\Delta(r)}\left[C(x,p)r^2\exp\left(-\frac{2rF^2_{,\gamma}}{C(x,p)}\right) - \Delta(r)\right]^{-1/2} \\ &\quad \times \left\{2C(x,p)r^3\left[\Delta(r)F^2_{,\gamma} + C(x,p)\frac{\partial\Delta(r)}{\partial r}\right] + \right. \qquad (14) \\ &\quad \times \Delta(r)\exp\left(\frac{2rF^2_{,\gamma}}{C(x,p)}\right)\left[2\Delta(r)\left(C(x,p) - 2rF^2_{,\gamma}\right)\right. \\ &\quad \left.\left. - 3C(x,p)r\frac{\partial\Delta(r)}{\partial r}\right]\right\}.\end{aligned} \tag{15}$$

Therefore, the evolution of the quantized geodesic congruence, $\tilde{\Theta}$, is given as

$$\begin{aligned}\frac{d\tilde{\Theta}}{d\tau} &= \frac{\exp\left(\frac{-2rF^2_{,\gamma}}{C(x,p)}\right)}{4C^2(x,p)r^4\Delta^2(r)\left[\Delta(r)\exp\left(\frac{2rF^2_{,\gamma}}{C(x,p)}\right) - C(x,p)r^2\right]} \\ &\quad \times \left\{-\frac{\Delta^2(r)}{\exp\left(-\frac{4rF^2_{,\gamma}}{C(x,p)}\right)}\left[8\Delta^2(r)\left(C(x,p) - rF^2_{,\gamma}\right)\right.\right. \\ &\quad - \frac{3C(x,p)r^2}{\left[\Delta'(r)\right]^{-2}} + 2r\Delta(r)\left(\Delta'(r)\left(2rF^2_{,\gamma} - 4C(x,p)\right)\right. \\ &\quad \left.\left. + 3C(x,p)r\Delta''(r)\right)\right] \\ &\quad + \frac{2r^2 C(x,p)\Delta(r)}{\exp\left(-\frac{2rF^2_{,\gamma}}{C(x,p)}\right)}\left[\Delta^2(r)\left(6C(x,p) - 8rF^2_{,\gamma}\right)\right. \\ &\quad - \frac{3C(x,p)r^2}{\left[\Delta'(r)\right]^{-2}} + r\Delta(r)\left(\Delta'(r)\left(2rF^2_{,\gamma} - 6C(x,p)\right)\right. \\ &\quad \left.\left. + 5C(x,p)r\Delta''(r)\right)\right] \\ &\quad + 4C(x,p)r^6\left[\frac{\Delta^2(r)}{\left(F^2_{,\gamma}\right)^{-2}} + C(x,p)\Delta(r)\Delta'(r)F^2_{,\gamma}\right. \\ &\quad \left.\left. + C^2(x,p)\left(\left[\Delta'(r)\right]^2 - \Delta(r)\Delta''(r)\right)\right]\right\}.\end{aligned} \tag{16}$$

As discussed, the Finsler/Hamilton structure F possesses an uppermost importance in the proposed quantization ansatz, as it allows for the integration of quantum revisions and the derivation of the corresponding metric, where quantum-induced revisions are incorporated. The question that arises is which expression can be designated to F? For the sake of simplicity, let us assume the most straightforward formulation provided by the Klein metric [54,55], which is a metric on the complement of a fixed quadric in a projective space, defined using a cross-ratio. Since it is utilized to provide metrics in hyperbolic, elliptic, and Euclidean geometries, it is appropriate to realize a unifying concept in geometry, gravity and electromagnetism [24].

The numerical assessments of both $d\Theta/d\tau$, as analytically expressed in Eq. (11), and $d\tilde{\Theta}/d\tau$, Eq. (16), are provided in Sect. 3.1. We still need to investigate whether the possible additional curvatures are artifacts that have emerged due to a specific coordinate transformation either known or unknown. Therefore, the quadratic invariant Kretschmann scalar is derived in Sect. 2.2.

2.2 Intrinsic quantum curvature

Regarding the additional curvatures, it is reasonable to argue that these curvatures could be artifacts in certain coordinate systems. Firstly, let us recall that these curvatures appear at quantum scales. Secondly, their nature is markedly different from the traditional ones. Thirdly, they are merely exposed through $\tilde{g}_{\mu\nu}$. The conventional $g_{\mu\nu}$, on which traditional GR is based, has semiclassical approximations that render the additional curvatures undetectable. For example, both the additional curvatures and the corresponding affine connections, along with further geometric structures, may or may not arise solely in the spherical coordinates which are utilize in the present calculations. Now, the approach to assess if the additional curvatures, for instant, are real, intrinsic, and essential consists of analyzing the invariance in every coordinate transformation. In this regard, two invariant scalars can be emphasized. The Ricci scalar, which is well-known for its role in characterizing vacuum solutions, is vanishing throughout. In contrast, the Kretschmann invariant scalar [56] may either be vanishing (implying that the curvatures could be mere artifacts and can be removed) or finite (implying that the curvatures are real, intrinsic, and essential, and thus cannot be eliminated). Consequently, it is important to emphasize that the Kretschmann scalar quantitatively assesses the degree of curvature of spacetime, independent of the coordinate system [41,42,56]. Moreover, the Kretschmann scalar K is identified among the fundamental polynomial curvature invariant scalars in GR, which are formulated as a sum of the squares of the components of the Riemann curvature tensor, namely, $R^{\alpha\beta\mu\nu}$ and $R_{\alpha\beta\mu\nu}$, for example,

$$K=\sum_{\gamma=0}^{3}\sum_{\beta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}R^{\gamma\beta\mu\nu}R_{\gamma\beta\mu\nu}. \tag{17}$$

It is clear that Eq. (17) illustrates that the calculation of K requires significant algebraic derivations. This procedure entails deriving the affine connections, which in turn enables us to derive the Riemann curvature tensor. When applying this to the charged, massive, non-rotating, and spherically symmetric Reissner–Nordström black hole, one finds that

$$\begin{aligned}K=\frac{1}{8r^8}\Bigg\{&\left(8Mr-\frac{3Q^2}{\pi\varepsilon}\right)^2+\frac{2}{\pi^2\varepsilon^2}\left(Q^2-8\pi\varepsilon rM\right)^2\\&+\frac{2}{\varepsilon^2}\left(1+\frac{3}{\pi^2}\right)\left(Q^2-4\pi\varepsilon rM\right)^2\\&+\frac{\left[3Q^4+4\pi\varepsilon r\left(Q^2\left[3r-8M\right]+8M\varepsilon r\left[2M-r\right]\right)\right]^2}{\varepsilon^2\left[Q^2+4\pi\varepsilon r\left(r-2M\right)\right]^2}\Bigg\}.\end{aligned} \tag{18}$$

We can now proceed with the same derivation process for $\tilde{g}_{\mu\nu}$. This leads us to a straightforward derivation of the quantized linear affine connections, detailed in Eq. (A4), and consequently, the quantized Riemann curvature tensor. It is apparent that Eq. (17) is also valid for the quantized metric, Eq. (1), so that

$$\tilde{K}=\sum_{\gamma=0}^{3}\sum_{\beta=0}^{3}\sum_{\mu=0}^{3}\sum_{\nu=0}^{3}\tilde{R}^{\gamma\beta\mu\nu}\tilde{R}_{\gamma\beta\mu\nu}. \tag{19}$$

The relevant Kretschmann scalar, Eq. (19), can be quantized in a straightforward fashion as

$$\begin{aligned}\tilde{K}=\frac{1}{256}\Bigg\{&\frac{16C(x,p)}{\pi^2\varepsilon^2r^8}\left(Q^2-4\pi\varepsilon rM\right)^2\left(1+\sin^4\theta\right)\\&+\frac{256C(x,p)}{r^4}\left[\frac{4r+8rM-\frac{Q^2}{\pi\varepsilon}}{4r^2}+\cot\theta+F_{,\gamma}^2\right]^2\\&+256C(x,p)\Delta^2(r)\left[\frac{1}{r}+\cot\theta+\frac{F_{,\gamma}^2}{C(x,p)}\right.\\&\left.+\frac{Q^2-4\pi\varepsilon rM}{r^2\left[Q^2+4\pi\varepsilon r\left(r-2M\right)\right]}\right]^2\\&+\frac{4096\pi^2\varepsilon^2r^4C(x,p)}{\left[Q^2+4\pi\varepsilon r\left(r-2M\right)\right]^2}\left[\frac{1}{r}+\cot\theta+\frac{F_{,\gamma}^2}{C(x,p)}\right.\\&\left.-\frac{\left(Q^2-4\pi\varepsilon rM\right)\left(Q^2+4\pi\varepsilon r\left[r-2M\right]\right)}{16\pi\varepsilon^2r^6}\right]^2\\&+\frac{16}{\pi^2\varepsilon^2r^6C(x,p)}\Big[C(x,p)\left(Q^2+4\pi\varepsilon\left[r^2+r-M-2rM\right]\right)\\&-2\pi\varepsilon rF_{,\gamma}^2\left(1-2M+2r\right)\Big]^2\\&+\frac{16}{\varepsilon^2r^8\left(Q^2+4\pi\varepsilon\left[r-2M\right]\right)^2}\Big[C(x,p)\Big(Q^4\end{aligned}$$

$$
\begin{aligned}
&+4\pi\varepsilon r Q^2\left[r-3M\right]\Big)\Big]\\
&-16\pi\varepsilon^2 r^2\left(r^3+\pi M\left[r-2M\right]\right)\Big)-8\pi\varepsilon^2 r^6 F_{,\gamma}^2\Big]^2\\
&+\frac{Q^2+4\pi\varepsilon\left[r-2M\right]}{\pi^3\varepsilon^3 r^{10}}\Big[-2C(x,p)Q^2\left[r-3\right]\\
&+8\pi\varepsilon r C(x,p)M\left[r-2\right]+r^2F_{,\gamma}^2\left(Q^2+4\pi\varepsilon r\left[r-2M\right]\right)\Big]^2\\
&+\frac{16}{\pi^2\varepsilon^2 r^{12}C(x,p)}\Big[r^2C(x,p)\Big(Q^2\\
&-8\pi\varepsilon r M+4\pi\varepsilon r\csc^2\theta\Big)+2\pi\varepsilon F_{,\gamma}^2\Big(r^4\csc^2\theta\\
&+2\cot\theta\left[r^4-\csc^4\theta\right]\Big)\Big]^2\\
&+\frac{8\left(Q^2-4\pi\varepsilon r M\right)}{\pi^2\varepsilon^2 r^8}\Big[2C(x,p)Q^2\left[r-1\right]\\
&+8\pi\varepsilon r C(x,p)\left(3M-2r\left[1+M\right]+r^2\right)\\
&+r^2F_{,\gamma}^2\left(Q^2+4\pi\varepsilon r\left[r-2M\right]\right)\Big]\\
&+\frac{16\left(Q^2-4\pi\varepsilon r M\right)}{\pi^2\varepsilon^2 r^8}\Big[C(x,p)Q^2\left[1+r\right]\\
&-4\pi\varepsilon r C(x,p)\left[M+r^2\right]-2\pi\varepsilon r^2F_{,\gamma}^2\csc^2\theta\Big]\\
&+\frac{64C(x,p)\Delta(r)}{\pi\varepsilon r^4\left(Q^2+4\pi\varepsilon r\left[r-2M\right]\right)^3}\Big[2\frac{r}{C(x,p)}F_{,\gamma}^2\Big(Q^4\\
&+4\pi\varepsilon r Q^2\left[r-3M\right]+4\pi^2\varepsilon^2 r^2\left[8M^2-4rM+r^3\right]\Big)\\
&+\frac{1}{\varepsilon\left(Q^2+4\pi\varepsilon r\left[r-2M\right]\right)}\Big[\pi\varepsilon^2\Big(-Q^2\\
&+4\pi\varepsilon r\left[r-2M\right]\Big)\Big(5Q^4+12\pi\varepsilon r Q^2\left[r-4M\right]\\
&-32\pi^2\varepsilon^2 r^2 M\left[r-3M\right]\Big)\\
&+8\pi\varepsilon^2\Big(Q^6+\pi\varepsilon r Q^4\left[7r-18M\right]\\
&-8\pi^2\varepsilon^3 r^3 M\Big[r^2\left(2+r+2\pi\right)+4M^2\left(2+3\pi\right)\\
&-2rM\left(4+5\pi\right)\Big]\Big)\\
&+2\pi\varepsilon^2 r^2 Q^2\Big(r^2\left[r+6\pi\right]+M^2\left[4+46\pi\right]\\
&-2M\left[r+17\pi r\right]\Big)\Big]\Big]^2\Big\}.
\end{aligned}
\tag{20}
$$

The numerical analyses of K, which is derived using $g_{\mu\nu}$, Eq. (18), and $\tilde{K}$, which is derived using $\tilde{g}_{\mu\nu}$, Eq. (20), are detailed in Sect. 3.2.

3 Numerical analyses and discussions

Following the derivation of the different analytical expressions, we now proceed to their numerical solutions, which fundamentally assist in drawing definitive conclusions. We begin with $d\Theta/d\tau$, Eq. (11), and subsequently advance to $d\tilde{\Theta}/d\tau$, Eq. (16). The numerical evolution is greatly dependent on the conformal coefficient $C(x,p)$, Eq. (3). As discussed, $C(x,p)$ combined all quantum operators and quantities imposed on the metric tensor and thereby in GR. As we focus on evaluating the space singularity, the other independent factor for the numerical analyses is the radial distance r. The subsequent section explores the relevant numerical results.

3.1 Timelike geodesic congruence with $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$

Using approximate dummy values for the quantities $M = 0.54 \pm 0.03$, $Q = 0.22 \pm 0.05$, $F_{,\gamma}^2 = 0.22 \pm 0.001$, and $\varepsilon = 0.01 \pm 0.0005$, Appendix B, the numerical analyses regarding the timelike geodesic congruence in charged, massive, non-rotating, and spherically symmetric Reissner–Nordström black hole are executed. The suggested values are merely *ad hoc* inputs. There is no theoretical basis neither supporting not opposing them. In the absence of any other alternatives, such approximations are permissible. Thus, an approximate estimation appears to be inevitable. It may enable the drawing of qualitative conclusions. Appendix B provides details concerning the correlation between Q and M at fixed $F_{,\gamma}^2$ and ε, and also the opposite situation. The primary motivation for the chosen set of parameters is to identify the optimal variations that affect the results for $d\Theta/d\tau$, $d\tilde{\Theta}/d\tau$, K and $\tilde{K}$. The mathematical constraints involved in defining the different quantum operators should be addressed elsewhere. The evolution of timelike geodesic congruence in a gravitationally collapsing spacetime is a particular phenomenon in which the initial conditions of the congruence expansion, rotation, and shear is assumed to affect the overall behavior and focusing characteristics of the trajectories. Analyzing this evolution is essential for comprehending the dynamics of spacetime and the conduct of physical systems contained within it.

In the top panel of Fig. 1, the evolution of the timelike geodesic congruence, Eq. (11), is shown in relation to the radial distance r and $C(x,p) = 1$ obtained with the conventional metric $g_{\mu\nu}$. The dependence on $C(x,p)$ is kept to merely provide an illustrative comparison of the results when quantization is taken into account. On the other hand, the dependence on r is distinctive. As r declines, $d\Theta/d\tau$ decreases exponentially. At extremely small values of r, the negative values of $d\Theta/d\tau$ significantly monotonically increase. The results undoubtedly relate to the existence of negative curvature almost universally [56]. This result points to a large trajectory congruence, which is generated from the velocity fields. In this regard, we recall that the negative evolution of geodesic congruence is assumed to refer to the behavior of a family of trajectories within a spacetime manifold that departs from their expected trajectories due to multiple factors, including curvature, nonstaticity, and spatial inhomogeneity [9,57]. This phenomenon may lead to a finite increase in the expansion scalar before focusing occurs, and therefore point to a nonmonotonic behavior of the congruence. The results are likely to be significantly influenced by the proposed set of parameters; however, maintaining con-

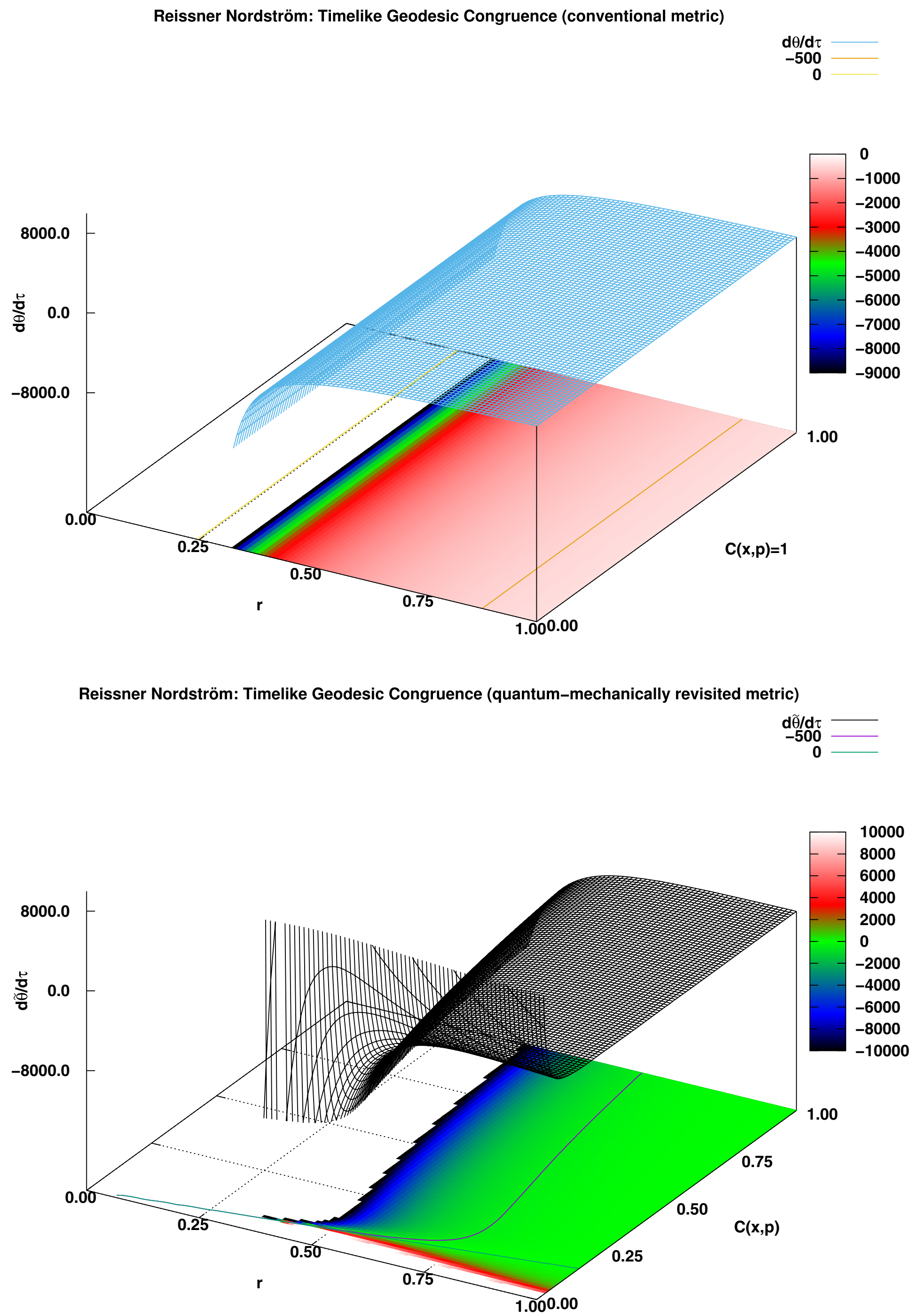

Fig. 1 The evolution of geodesic congruence as a function of the radial distance r and the conformal coefficient $C(x, p)$, Eq. (11). The top panel presents the numerical results for $d\Theta/d\tau$ derived with $g_{\mu\nu}$, i.e., $C(x, p) = 1$. The bottom panel provides the numerical assessments for $d\tilde{\Theta}/d\tau$ as indicated in Eq. (16) using $\tilde{g}_{\mu\nu}$

sistency throughout all calculations facilitates a systematic comparison.

The bottom panel of Fig. 1 displays the same as the top panel of Fig. 1. Here, the evolution of the quantized geodesic congruence, $d\tilde{\Theta}/d\tau$, Eq. (16). Notably, there is a significant dependence on r, as the values of $d\tilde{\Theta}/d\tau$ alternate between positive and negative signs. In this regard, let us recall that the positive evolution of geodesic congruence likely relates to the occurrence in which the shear and expansion of geodesic congruence augment over time. This leads to a focusing effect on the congruence. The dependence of $d\tilde{\Theta}/d\tau$ on $C(x, p)$ is also phenomenal. At small $C(x, p)$, the numerical values of $d\tilde{\Theta}/d\tau$ approach positive and negative infinity depending on r. Obviously, this refers to a large trajectory congruence, which is generated from the velocity fields.

We also find that $d\Theta/d\tau$, top panel, and $d\tilde{\Theta}/d\tau$, bottom panel, are mostly distinguishable. Furthermore, one finds that as $C(x, p)$ approaches unity, the values of $d\tilde{\Theta}/d\tau$ come closer to that of $d\Theta/d\tau$. We remark that even at $C(x, p) = 1$, both quantities are not fully identical. A reasoning can be

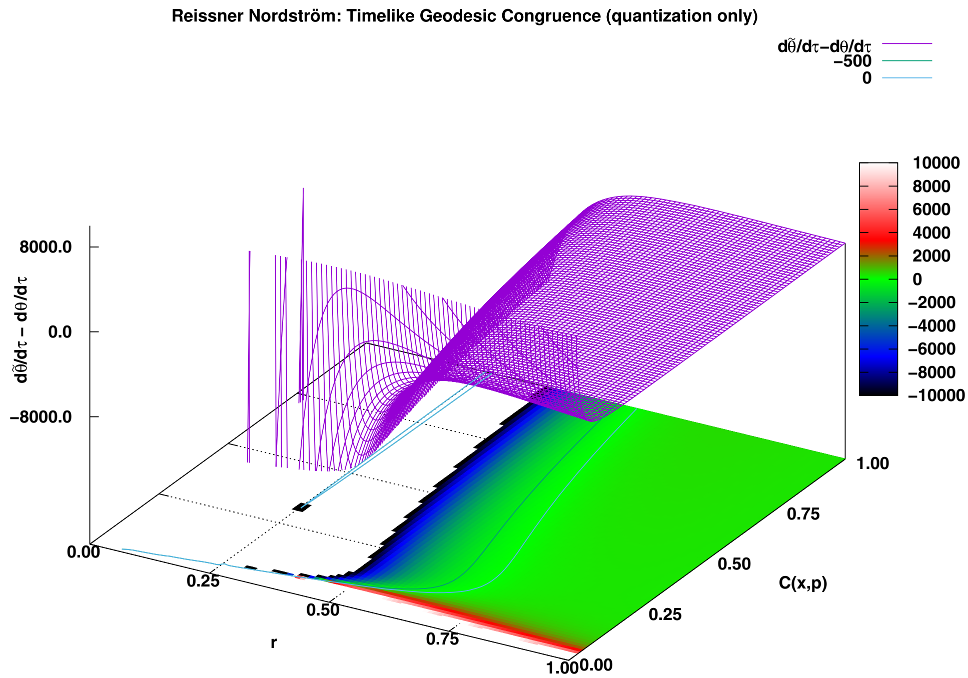

Fig. 2 Identical to Fig. 1, but here the figure emphasizes exclusively the contributions from pure quantization. The difference $d\tilde{\Theta}/d\tau - d\Theta/d\tau$ is specifically associated with the evolution of timelike geodesic congruence related to the curvatures arising from the proposed geometric quantization ansatz

realized from the quantization procedure which was briefly introduced in Sect. 1. This is much more diverse than just the conformal transformation in Eq. (1).

The additional contributions to $d\Theta/d\tau$ resulting from the proposed geometric quantization are shown in Fig. 2, namely $d\tilde{\Theta}/d\tau - d\Theta/d\tau$. It is clear that the geometric quantization, which suggests a conformal transformation of $g_{\alpha\beta}$ as merely one component, results in a substantial timelike geodesic congruence. A large nonmonotonic timelike geodesic congruence occurs, influenced by varying r and $C(x, p)$. In relation to $d\tilde{\Theta}/d\tau$, $d\Theta/d\tau$ is quite small. This indicates that GR at the quantum scale seems to expose a rich spacetime structure that is evidently hidden at the semiclassical scale. A quantitative evaluation of the evolution of timelike geodesic congruence will be derived in another context, especially when the mathematical challenges discussed in Section 2 are resolved.

As previously mentioned, the observation of extensive timelike geodesic congruence prompts the inquiry of whether the associated curvatures are genuinely real or simply artifacts that arise in certain coordinate systems. The subsequent section aims to address this inquiry. The invariant scalars of GR provide an evaluation of the essentiality of the additional (quantum) curvatures.

3.2 Kretschmann invariant scalar with $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$

The significance of the Kretschmann invariant scalar in highlighting the intrinsic quantum curvatures is crucial. First, it provides an algebraic representation of the curvature of spacetime. Second, it is independent of any coordinate system. This independence facilitates the characterization of the additional curvatures, particularly in determining whether they are mere artifacts or genuinely real, essential, and intrinsic. The artifact curvature, which per definition are inessential and extrinsic, seemingly manifests in certain coordinate systems and is expected to dissipate in others. They can be removed through specific coordinate transformations. The inquiry then becomes: which transformations are relevant? It is apparent that the currently recognized set of coordinate systems and transformations cannot be confined to a closed set. This suggests that any evaluation of additional curvatures based solely on coordinate transformations would not be able to differentiate between real and artifact curvatures. The reason for this is the potential existence of other coordinate systems and transformations that have not yet been discovered, in which the curvatures may be considered removable artifacts or intrinsic real.

In Fig. 3, the numerical results of the Kretschmann invariant scalar are displayed as a function of r and $C(x, p)$. The same parameters that were used in Figs. 1 and 2 are taken into account here, Appendix B. We analyze the differences among $\tilde{K}$ (geometrically quantized GR), K (Conventional GR), and $\tilde{K} - K$ (purely quantized GR). We observe that K (conventional $g_{\alpha\beta}$ and thereby GR) significantly rises as r decreases (dotted lattice). It seems that there is almost no influence from $C(x, p)$. A nearly identical observation is made regarding the dependence of $\tilde{K}$ ($\tilde{g}_{\alpha\beta}$ and thereby quantized GR) on r and $C(x, p)$ (dashed lattice). For large values of r, both K and $\tilde{K}$ are approaching zero, i,e., no curvatures. As r decreases, the values of K and $\tilde{K}$ gradually become positive. The positive values imply that the curvatures of the underlying spacetime, specifically in the context of a Reissner–Nordström black hole, are positive. These positive values indicate that the black hole does not represent a singularity, but instead signifies a region characterized by positive curvature. On the other hand, this result implies that the corresponding curvatures are real, essential, and intrinsic.

Regarding the contributions from pure quantization, that are $\tilde{K} - K$, a decrease in r greatly diminishes the values of $\tilde{K} - K$. For the moment, let us set aside the negative values that denote negative curvatures in spacetime and focus on the nonvanishing values of $\tilde{K} - K$, which refer to real, essential, and intrinsic curvatures, i.e., the additional curvatures which arise at quantum scales and therefore are related to the emergence of additional sources of quantum gravity. The latter seems to be left undisclosed in conventional GR. While the classical approximations applied to GR exhibit notable accuracy at large scales, they seem to fully mask quantum phenomena. In this regards, the negative values are especially important in the context of rotating black holes [58–63], where the presence of electric charge can induce a negative curvature. If the negative curvature observed in the Reissner–Nordström black hole proves to be valid, it could suggest a more complex inner structure, which seems to manifest at quantum scales.

4 Conclusions

We analyzed the timelike geodesic congruence of the Reissner–Nordström metric, utilizing the conventional $g_{\mu\nu}$ and quantized metric $\tilde{g}_{\mu\nu}$. While $g_{\mu\nu}$ characterizes smooth continuous pseudo-Riemann geometry, $\tilde{g}_{\mu\nu}$ is applied to quantized pseudo-Riemann geometry. We find that the evolution of the geodesic congruence expansion is non-zero in the gravitational field of the charged, massive, non-rotating, and spherically symmetric Reissner–Nordström black hole. Specifically, a reduction in radial distance results in an extremely large evolution of the geodesic congruence expansion. This monotonic dependence is largely determined by the approximate values assigned to the parameters M (mass), Q (electric charge), and ε (electric constants). The proposed quantization evidently enhances the evolution of the geodesic congruence expansion and imposes a significant dependence on the radial distance r and the general conformal quantization factor $C(x, p)$. In this context, the rate of geodesic congruence expansion rapidly switches between large positive and large negative values, especially at small $C(x, p)$.

It is important to note that the proposed geometry quantization maintains the main postulates of GR. The conventional metric, affine connections, geodesic equations, and Riemann curvature tensors can be entirely recovered, at $C(x, p) = 1$. Consequently, the evolution of the geodesic congruence expansion with the quantized metric aligns with the outcomes expected from the conventional metric. This enables the assessment of the contributions arising from the proposed quantization. Notably, the dependence on the radial distance to the singularity is significant. As one approaches the singularity, the evolution becomes substantial and abruptly switches between large positive and large negative values. In comparison to the classical geodesic congruence expansion evolution results, the additional quantum contributions indicate that GR appears to possess a complex spacetime structure at the quantum level, which is often overlooked in the classical framework.

To evaluate whether the emerged additional curvatures are real, essential, and intrinsic, we conducted an analysis of the Kretschmann invariant scalar K. We discovered that the values of K obtained from $g_{\mu\nu}$ and the values of $\tilde{K}$ derived from $\tilde{g}_{\mu\nu}$ remain finite everywhere, especially at small radial distances r. These values increase significantly as one approaches the singularity, i.e., as r decreases. Thus, we conclude that the additional curvatures obtained from both versions of the fundamental metric are real, essential, and intrinsic. Furthermore, we observe that quantization locally enhances the corresponding curvatures.

We conclude that the curvature present in charged, massive, non-rotating, and spherically symmetric Reissner–Nordström black holes is likely influenced by additional sources of gravitation originating from quantum mechanically imposed revisions, which may modify the geometry of the black hole, particularly within the relativistic and quantum domains.

Author contributions The responsibility for proposing the conception of the present study lies with AT, who also undertook the tasks of designing and managing the research, interpreting the results, and preparing the manuscript. SOA was responsible for proposing physical interpretations and writing the draft. AAA, SGE and MN contributed to the writing and proofreading of the manuscript. The final version of the manuscript was unanimously approved by all authors.

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Declarations

Conflict of interest The authors declare that there are no conflict of interest regarding the publication of this published article!

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Appendix A: Geodesic equations

As stated in Eq. (1), $\tilde{g}_{\mu\nu}$ exhibits a linear dependence on $g_{\mu\nu}$. Consequently, the proper time can be expressed from $-c^2 d\tau^2 = ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. It is clear that this holds true for the proposed geometric quantization of the metric tensor $g_{\mu\nu}$. Therefore, in natural units, we find that $-d\tilde{\tau}^2 = d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu$ and

$$\tilde{\tau}_{ab} = \int_0^1 \sqrt{-\tilde{g}_{\mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta}} = \int_0^1 \mathcal{L}\left(\frac{dx^\mu}{d\zeta}, x^\mu\right) d\zeta. \quad \text{(A1)}$$

In this regard, Euler–Lagrange equations read

$$-\frac{d}{d\zeta}\frac{\partial\mathcal{L}}{\partial(dx^\gamma/d\zeta)} + \frac{\partial\mathcal{L}}{\partial x^\gamma} = 0, \quad \text{(A2)}$$

where $\mathcal{L}$ is the Lagrangian. Accordingly, the quantized geodesic equations can be given as follows:

$$\frac{d^2x^\mu}{d\tau^2} + \tilde{\Gamma}^\mu_{\delta\nu}\frac{dx^\delta}{d\tau}\frac{dx^\nu}{d\tau} = -\frac{g_{\delta\nu}}{2\tilde{g}_{\mu\gamma}}F^2_{,\gamma}C(x,p)\frac{dx^\delta}{d\tau}\frac{dx^\nu}{d\tau}, \quad \text{(A3)}$$

where the function $C(x,p)$ is defined in Eq. 3, while F^2 denotes the structure within the tangent or cotangent bundle. As shown in Eq. (A3), the proposed geometric quantization adds an additional term that encompasses portion of quantum operators and quantities. Also, the quantized affine connections, $\tilde{\Gamma}^\mu_{\delta\nu}$ comes up with another term,

$$\tilde{\Gamma}^\mu_{\delta\nu} = \Gamma^\mu_{\delta\nu} + \frac{F^2_{,\gamma}}{2C(x,p)}\left(\delta^\mu_\nu + \delta^\mu_\delta - g^{\mu\gamma}g_{\delta\nu}\right). \quad \text{(A4)}$$

For the sake of simplicity, let us assume that the simplest Finster metric, namely Klein metric [54,55], expresses the corresponding structure. The Klein metric, also as the Beltrami–Klein model of hyperbolic geometry [64], represents hyperbolic space situated within the Euclidean unit ball, where geodesics are represented as straight line segments [65]. This feature establishes the Klein metric as one of the most fundamental and simple examples of a projectively flat Finsler metric. Within Finsler geometry, the Klein metric is of paramount importance for several reasons [66]:

- it is locally projectively flat, which means that geodesics are straight lines,
- it maintains a constant flag curvature of -1, analogous to sectional curvature in Finsler geometry, and
- it serves as the fundamental metric from which the Funk–Hilbert metrics [67] can be derived through deformation with a $1-$form.

Then, the structure in Hamilton geometry, where 2-homogeneity in $\dot{x}_0^\mu$ and positive mass m allow to identify $m\dot{x}_0^\mu$ as $\dot{p}_0^\mu$, can be expressed as [68]

$$F^2_{,\gamma} = 2\frac{\langle x_0^\mu \cdot p_0^\nu\rangle}{\left[\left(x_0^\mu\right)^2 - 1\right]^2}\left\{\left[1-\left(x_0^\mu\right)^2\right]p_0^\nu - x_0^\mu\langle x_0^\mu \cdot p_0^\nu\rangle\right\}. \quad \text{(A5)}$$

Both analytical and numerical analyses presented herein concentrate on the contributions provided by the proposed geometric quantization and their effects on space singularity as well as the intrinsic curvatures that arise at relativistic (quantum) scales. Specifically, we examine the additional contributions to the timelike geodesic congruence in the Reissner–Nordström black hole. The primary concept applied involves a thorough comparison between conventional and quantized timelike geodesic congruence.

Appendix B: Parameter study

The numerical analyses are executed with respect to r and $C(x,p)$. On the other hand, there is a range of parameters, including M (mass), Q (electric charge), and ε (vacuum permeability or electric and magnetic constants) associated with the Reissner–Nordstr"om black hole, in addition to $F^2_{,\gamma}$, the derivative of the Hamilton structure, which also affects the numerical results. To determine the appropriate values for each parameter, we have undertaken a comprehensive parameter study aimed at identifying the optimal variations within the entire set of parameters that yield the best results for the complete set of $d\Theta/d\tau$, $d\tilde{\Theta}/d\tau$, K, and $\tilde{K}$. The optimal results are evaluated based on the simultaneous physical behavior of these four quantities.

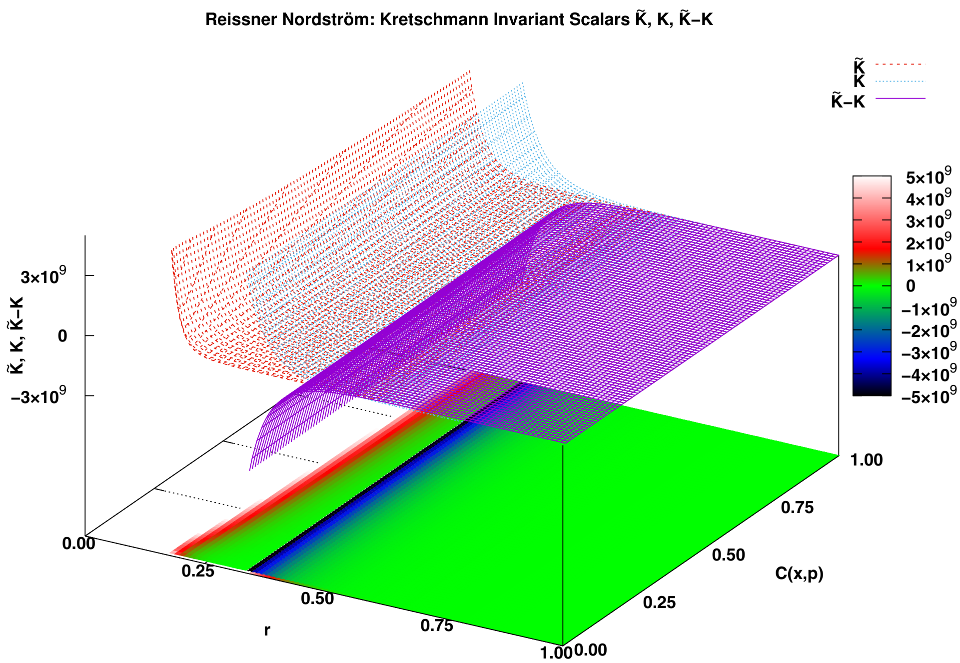

Fig. 3 The Kretschmann invariant scalar is presented as a function of the radial distance r and the generic conformal quantization factor $C(x, p)$, Eq. (1). The results obtained for $g_{\alpha\beta}$ are compared with the results for $\tilde{g}_{\alpha\beta}$. A compare between $\tilde{K}$ (dashed lattice), K (dotted lattice), and $\tilde{K} - K$ (solid lattice) is presented

The optimal results for $d\Theta/d\tau$, $d\tilde{\Theta}/d\tau$, K, and $\tilde{K}$ are illustrated in Figs. 1, 2 and 3. All other attempts have been excluded. Figure 4 illustrates the relationships between M and Q, as well as the constants ε and $F^2_{,\gamma}$ (top panel). These analyses should be conducted in a self-correlated manner. The bottom panel shows the correlations between ε and $F^2_{,\gamma}$ while keeping M and Q constant. From Fig. 4, we find that $M = 0.54 \pm 0.03$, $Q = 0.22 \pm 0.05$, $F^2_{,\gamma} = 0.22 \pm 0.001$, and $\varepsilon = 0.01 \pm 0.0005$.

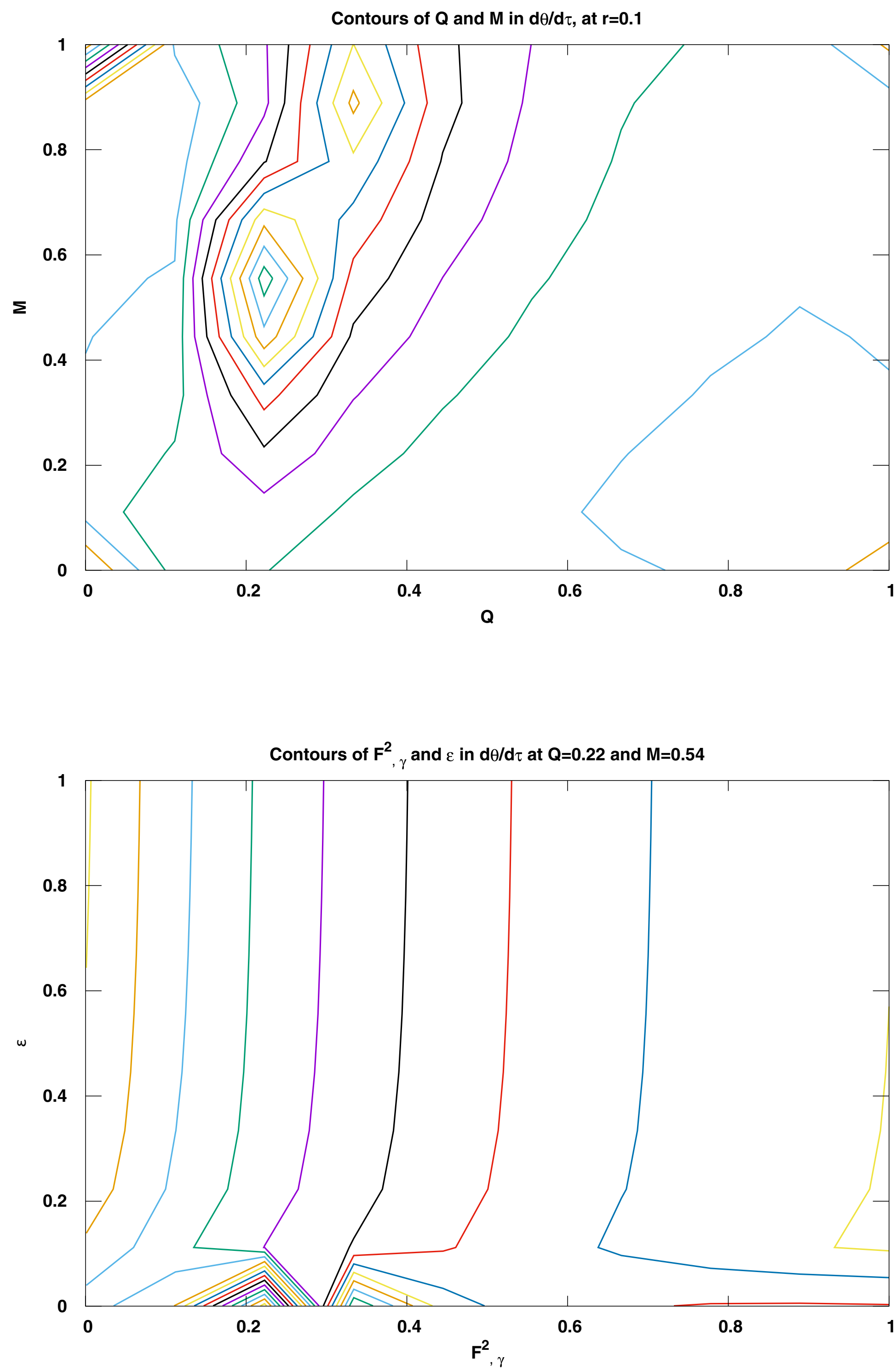

Fig. 4 The parameter analysis of M and Q was carried out at a constant ε and $F^2_{,\gamma}$ (top panel), and similarly for ε and $F^2_{,\gamma}$ while keeping M and Q constant (bottom panel). The full set of 4 parameters allows for the determination of results for $\Theta/d\tau$, $d\tilde{\Theta}/d\tau$, K, and $\tilde{K}$, which are represented in Figs. 1, 2 and 3

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